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q-Trinomial coefficients and the dilute A model

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Abstract. We express the configuration sums of the dilute A_L model in regimes 1^+ and 2^+ in terms of q-trinomial coefficients. We then extend previous results to give expressions for all phases and regimes, for odd L, for finite system size.

1. Introduction

In recent years, solvable lattice models have proved to be the source of new q-series identities of Rogers–Ramanujan type [1, 2]. Polynomial expressions arising in the configuration sums of models, before the infinite lattice limit is taken, provide a natural 'finitization' of the bosonic side of the identities [3, 4]. Finitization is the basis for one method of their proof [5]. In what follows we show how q-trinomial coefficients provide a natural and elegant way to write the configuration sums of the dilute A models [6], originally stated in workable, but cumbersome, form in [7].

The q-trinomial coefficients were introduced by Andrews and Baxter [8] to express the solutions of corner transfer matrix calculations for lattice-gas generalizations of the hard hexagon model. It was pointed out in [9] that q-trinomial coefficients are implicit in work on the dilute A_L models by Warnaar and Pearce [2, 10], who obtained Rogers–Ramanujan-type identities, the bosonic side of which came from particular configuration sums for $L = 3, 4, 6$. Indeed, the 'railroad' polynomials at the end of [11], which feature the q-trinomial coefficients, correspond to those for one regime of the dilute A model [12].

Here we make explicit the way in which the q-trinomial coefficients appear in the dilute A model configuration sums for L odd. We next present for the first time complete expressions for regimes 3^+ and 4^+, which previously were treated only for the subset of configurations corresponding to the ground states, and then only in the thermodynamic limit. Finally, we consider the process of taking this limit for all of our expressions.

2. The Trinomials

The *trinomial coefficient* $\binom{n}{j}_2$ is the coefficient of x^{j+n} in the expansion

$$(1 + x + x^2)^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^{j+n} \tag{2.1}$$

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so that

$$\binom{n}{j}_2 = \sum_{k\geqslant 0} \frac{n!}{k!(k+j)!(n-j-2k)!} \tag{2.2}$$

satisfying the recurrence

$$\binom{n}{j}_2 = \sum_{k=-1}^{1} \binom{n-1}{j+k}_2 . \tag{2.3}$$

The *Gaussian polynomials* $\left[{m \atop k}\right]$, also called the q-binomial coefficients, and the *Gaussian multinomials* $\left[{m \atop k,l}\right]$ are defined by [13]

$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{(q)_m}{(q)_k (q)_{m-k}} \qquad \begin{bmatrix} m \\ k,l \end{bmatrix} = \frac{(q)_m}{(q)_k (q)_l (q)_{m-k-l}} \tag{2.4}$$

where $(a;q)_n = (a)_n = \prod_{j=0}^{n-1}(1-aq^j)$.

Among a number of q-analogues of the trinomial coefficients given in [8] is

$$\begin{aligned} \binom{m;\, B;\, q}{A}_2 &= \sum_{k\geqslant 0} q^{k(k+B)} \frac{(q)_m}{(q)_k (q)_{k+A} (q)_{m-2k-A}} \\ &= \sum_{k\geqslant 0} q^{k(k+B)} \begin{bmatrix} m \\ k, k+A \end{bmatrix} . \end{aligned} \tag{2.5}$$

Hereafter, we will refer to the q-trinomial coefficients simply as 'trinomials'.

Various identities and recurrences for the trinomials are to be found in [8] (and further in [14, 15]); we give here those needed for our discussion of the dilute A models. We require only $B = A, A-1$, so that we can use the notation [9]

$$\binom{m;\, A-p;\, q}{A}_2 = \begin{bmatrix} m \\ A \end{bmatrix}_2^{(p)} \qquad p = 0, 1. \tag{2.6}$$

The following symmetries hold for $p = 0, 1$:

$$\begin{bmatrix} m \\ A \end{bmatrix}_2^{(0)} = \begin{bmatrix} m \\ -A \end{bmatrix}_2^{(0)} \qquad \begin{bmatrix} m \\ A \end{bmatrix}_2^{(1)} = q^A \begin{bmatrix} m \\ -A \end{bmatrix}_2^{(1)} . \tag{2.7}$$

The trinomials satisfy

$$\begin{bmatrix} m \\ A \end{bmatrix}_2^{(0)} = \begin{bmatrix} m-1 \\ A-1 \end{bmatrix}_2^{(0)} + q^{m-1} \begin{bmatrix} m-1 \\ A \end{bmatrix}_2^{(1)} + q^{m+A} \begin{bmatrix} m-1 \\ A+1 \end{bmatrix}_2^{(0)} \tag{2.8}$$

$$\begin{bmatrix} m \\ A \end{bmatrix}_2^{(1)} = \begin{bmatrix} m-1 \\ A-1 \end{bmatrix}_2^{(0)} + q^{m-1} \begin{bmatrix} m-1 \\ A \end{bmatrix}_2^{(1)} + q^{A} \begin{bmatrix} m-1 \\ A-1 \end{bmatrix}_2^{(0)} \tag{2.9}$$

$$\begin{bmatrix} m \\ A-1 \end{bmatrix}_2^{(0)} = \begin{bmatrix} m-1 \\ A-1 \end{bmatrix}_2^{(0)} + q^{m-1} \begin{bmatrix} m-1 \\ A \end{bmatrix}_2^{(1)} + q^{m-A+1} \begin{bmatrix} m-1 \\ A-2 \end{bmatrix}_2^{(0)} . \tag{2.10}$$

We point out that these, like (2.3), are depth one recurrences, with three terms on the right-hand side. They can be rewritten in several guises using the identity

$$\begin{bmatrix} m \\ A-1 \end{bmatrix}_2^{(0)} + q^m \begin{bmatrix} m \\ A \end{bmatrix}_2^{(1)} = \begin{bmatrix} m \\ A \end{bmatrix}_2^{(0)} + q^{m+1-A} \begin{bmatrix} m \\ A-1 \end{bmatrix}_2^{(1)} . \tag{2.11}$$

However, it was the following identity [8] which was the key to the simplification of the configuration sums appearing in [7]:

$$\begin{bmatrix} m \\ A \end{bmatrix}_2^{(1)} = \begin{bmatrix} m \\ A \end{bmatrix}_2^{(0)} + q^A (1-q^m) \begin{bmatrix} m-1 \\ A+1 \end{bmatrix}_2^{(0)} . \tag{2.12}$$

The q-trinomial coefficients have been generalized in the work of Warnaar [16] and Schilling [9] to q-multinomial coefficients $\left[\begin{smallmatrix} m \\ a \end{smallmatrix}\right]_n^{(p)}$. They provide the $n = 2$ case, and the $n = 1$ case gives the Gaussian polynomials.

3. The dilute A model

The dilute A model was defined in [6] and comprehensive discussion and results are to be found in [7]; here we summarize the features of interest for this work, referring readers to [7] for more details.

The dilute A_L model is an L state or 'height' model. There are eight regimes in the model parameters λ, u, p for which it can be solved; however, for L odd, negating p corresponds only to a relabelling of the heights, so it suffices to consider only four:

$$\begin{array}{llll} \text{regime } 1^+ & 0 < p < 1 & 0 < u < 3\lambda & \lambda = \frac{\pi}{4}\left(1 - \frac{1}{L+1}\right) \\ \text{regime } 2^+ & 0 < p < 1 & 0 < u < 3\lambda & \lambda = \frac{\pi}{4}\left(1 + \frac{1}{L+1}\right) \\ \text{regime } 3^+ & 0 < p < 1 & 3\lambda - \pi < u < 0 & \lambda = \frac{\pi}{4}\left(1 + \frac{1}{L+1}\right) \\ \text{regime } 4^+ & 0 < p < 1 & 3\lambda - \pi < u < 0 & \lambda = \frac{\pi}{4}\left(1 - \frac{1}{L+1}\right). \end{array} \tag{3.1}$$

Using corner transfer matrix techniques [17], the order parameters of the model for L odd were found in [7]. The local height probability $P^{bc}(a)$ is the probability of height a occurring at a site, given that the model is in the phase indexed by b and c. In all regimes there are ferromagnetic phases $c = b$, and in regimes 3^+ and 4^+ there are antiferromagnetic phases $c = b \pm 1$ as well.

If we define the (one-dimensional) configuration sums

$$X_m^{\sigma_1\sigma_{m+1}\sigma_{m+2}}(q) = \sum_{\sigma_2\ldots\sigma_m} q^{\sum_{j=1}^m jH(\sigma_j,\sigma_{j+1},\sigma_{j+2})} \tag{3.2}$$

the local height probabilities take the form

$$P^{bc}(a) = \lim_{m\to\infty} \frac{q^{-a^2\lambda/\pi} S(a) X_m^{abc}(q)}{\sum_{a=1}^L q^{-a^2\lambda/\pi} S(a) X_m^{abc}(q)} \tag{3.3}$$

in regimes 1^+ and 2^+, and in regimes 3^+ and 4^+

$$P^{bc}(a) = \lim_{m\to\infty} \frac{q^{a^2\lambda/\pi} S(a) X_m^{abc}(q)}{\sum_{a=1}^L q^{a^2\lambda/\pi} S(a) X_m^{abc}(q)}. \tag{3.4}$$

The variable q is related to $p = \exp(-\epsilon)$, the elliptic nome in the definition of the model weights, by

$$\begin{array}{ll} q = \mathrm{e}^{-12\pi\lambda/\epsilon} & \text{for regimes } 1^+ \text{ and } 2^+\text{, and} \\ q = \mathrm{e}^{-4\pi(\pi-3\lambda)/\epsilon} & \text{for regimes } 3^+ \text{ and } 4^+. \end{array} \tag{3.5}$$

The function $H(a, b, c)$, determined from the model's Boltzmann weights, is tabulated in appendix B of [7] for regimes 1^+ and 2^+. Due to a simple relation between the H functions, the form of the solution for regimes 3^+ and 4^+ can be obtained by replacing q with $1/q$ in the solution for regimes 2^+ and 1^+, respectively. (Having obtained the correct form, the meaning of q must be taken from (3.5).) The crossing factor $S(a)$ will not concern us.

The configuration sums $X_m^{abc}(q)$ obey a recurrence relation

$$X_m^{abc}(q) = q^{mH(b-1,b,c)} X_{m-1}^{ab-1b}(q) + q^{mH(b,b,c)} X_{m-1}^{abb}(q) + q^{mH(b+1,b,c)} X_{m-1}^{ab+1b}(q). \tag{3.6}$$

Again we point out that this is of depth one, with three right-hand side terms. This can be contrasted with the ABF [18] and CSOS [19] models, in which the adjacency rules force neighbouring heights to differ by one, so that the corresponding recurrence relations have only two terms on the right-hand side. The Gaussian polynomials (2.4), which themselves satisfy such a recurrence, are found in the solutions. In the CSOS model, X_m^{abc} has the form

$$\sum_{j=-\infty}^{\infty} q^{(\text{quadratic in } j)} \begin{bmatrix} m \\ (m/2) - \alpha j + \beta \end{bmatrix} \tag{3.7}$$

while for ABF, the difference of two terms with such structure is seen.

The solutions to (3.6) for regime 1^+ and 2^+ are listed as (5.26) and (5.36) of [7]. The solutions are expressed in terms of two functions $F_m^s(b)$ and $G_m^r(b)$, defined by:

$$\begin{aligned} F_m^s(b) = q^{(a-s)a/2} \sum_{j,k=-\infty}^{\infty} \Bigg\{ & q^{(L+2)(L+1)j^2+[(L+2)a-(L+1)s]j+k(k+2(L+1)j+a-b)} \\ & \times \begin{bmatrix} m \\ k, k+2(L+1)j+a-b \end{bmatrix} \\ & - q^{(L+2)(L+1)j^2+[(L+2)a+(L+1)s]j+as+k(k+2(L+1)j+a+b)} \\ & \times \begin{bmatrix} m \\ k, k+2(L+1)j+a+b \end{bmatrix} \Bigg\} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} G_m^r(b) = q^{(a-r)a/2} \sum_{j,k=-\infty}^{\infty} \Bigg\{ & q^{(L+1)Lj^2-[(L+1)r-La]j+k(k+2(L+1)j+a-b)} \\ & \times \begin{bmatrix} m \\ k, k+2(L+1)j+a-b \end{bmatrix} \\ & - q^{(L+1)Lj^2+[(L+1)r+La]j+ar+k(k+2(L+1)j+a+b)} \\ & \times \begin{bmatrix} m \\ k, k+2(L+1)j+a+b \end{bmatrix} \Bigg\}. \end{aligned} \tag{3.9}$$

Using trinomials (2.5) and (2.6), we see that we can immediately rewrite the function $F_m^s(b)$ with a single sum only:

$$\begin{aligned} F_m^s(b) = q^{(a-s)a/2} \sum_{j=-\infty}^{\infty} \Bigg\{ & q^{(L+2)(L+1)j^2+[(L+2)a-(L+1)s]j} \begin{bmatrix} m \\ 2(L+1)j+a-b \end{bmatrix}_2^{(0)} \\ & - q^{(L+2)(L+1)j^2+[(L+2)a+(L+1)s]j+as} \begin{bmatrix} m \\ 2(L+1)j+a+b \end{bmatrix}_2^{(0)} \Bigg\}. \end{aligned} \tag{3.10}$$

Four different expressions appear in the solution to the recurrences in regime 1^+ as obtained in [7]:

$$\begin{aligned} &\text{(i)} && F_m^b(b) \\ &\text{(ii)} && q^{b/2} F_m^{b+1}(b) \\ &\text{(iii)} && F_m^b(b) + (1-q^m) q^b F_{m-1}^{b+2}(b+1) \\ &\text{(iv)} && q^{b/2} F_m^{b+1}(b) + (1-q^m) q^{-b/2} F_{m-1}^{b-1}(b-1). \end{aligned} \tag{3.11}$$

As a consequence of (3.10), expressions (i) and (ii) have two terms of the same general structure as (3.7), with trinomials replacing the Gaussian polynomials. However, the other two expressions are not as 'neat' as those seen in other solvable models; they contain, in fact, six such terms, of order $m-1$ as well as order m.

Applying (2.12) to expressions (iii) and (iv), it becomes apparent that the fundamental building block in fact ought to be

$$\mathcal{F}_m^{s,p}(b) = q^{(a-b)(a+b-s)/2} \sum_{j=0}^{\infty} q^{(L+2)(L+1)j^2+[(L+2)a-(L+1)s]j} \begin{bmatrix} m \\ 2(L+1)j+a-b \end{bmatrix}_2^{(p)} \tag{3.12}$$

so that

$$F_m^s(b) = q^{b(b-s)/2}\{\mathcal{F}_m^{s,0}(b) - \mathcal{F}_m^{-s,0}(-b)\}. \tag{3.13}$$

Then each of the other two expressions in (3.11) can also be written as a difference of $\mathcal{F}$ functions:

$$\begin{aligned} &F_m^b(b) + (1-q^m)q^b F_{m-1}^{b+2}(b+1) = \mathcal{F}_m^{b+2,1}(b) - \mathcal{F}_m^{-b,1}(-b) \\ &q^{b/2}F_m^{b+1}(b) + (1-q^m)q^{-b/2}F_{m-1}^{b-1}(b-1) = \mathcal{F}_m^{b+1,1}(b) - \mathcal{F}_m^{-(b-1),1}(-b). \end{aligned} \tag{3.14}$$

For regime 2^+, the expressions found in the solution to (3.6) as given in [7] are

$$\begin{array}{ll} \text{(i)} & G_m^b(b) \\ \text{(ii)} & q^{-b/2}G_m^{b-1}(b) \\ \text{(iii)} & q^{-b/2}G_m^{b-1}(b) + (1-q^m)q^{b/2}G_{m-1}^{b+1}(b+1) \\ \text{(iii}^*\text{)} & q^{-1}G_m^1(2) + (1-q^m)qG_{m-1}^3(3) - q^{-1}G_m^1(0) \\ \text{(iv)} & G_m^b(b) + (1-q^m)q^{-b}G_{m-1}^{b-2}(b-1). \end{array} \tag{3.15}$$

We are led to define an analogous function

$$\mathcal{G}_m^{r,p}(b) = q^{(a-b)(a+b-r)/2} \sum_{j=-\infty}^{\infty} q^{(L+1)Lj^2-[(L+1)r-La]j} \begin{bmatrix} m \\ 2(L+1)j+a-b \end{bmatrix}_2^{(p)}. \tag{3.16}$$

Now, for the G_m^r function we have

$$G_m^r(b) = q^{b(b-r)/2}\{\mathcal{G}_m^{r,0}(b) - \mathcal{G}_m^{-r,0}(-b)\} \tag{3.17}$$

and the remaining expressions in (3.15) can be rewritten using (2.12) as

$$\begin{aligned} &q^{-b/2}G_m^{b-1}(b) + (1-q^m)q^{b/2}G_{m-1}^{b+1}(b+1) = \mathcal{G}_m^{b+1,1}(b) - \mathcal{G}_m^{-(b-1),1}(-b) \\ &G_m^b(b) + (1-q^m)q^{-b}G_{m-1}^{b-2}(b-1) = \mathcal{G}_m^{b,1}(b) - \mathcal{G}_m^{-(b-2),1}(-b). \end{aligned} \tag{3.18}$$

Applying (2.11) then simplifies (iii*) further.

Details of the substitutions and manipulations needed to obtain (3.14) and (3.18), as well as confirmation (using (2.8)–(2.10)) that the new forms we list below satisfy the original recurrence (3.6) were given in [20].

We are now in a position to rewrite the solutions from [7].

The possible values of b must be broken into four sets. For regime 1^+ they are

$$\begin{array}{ll} s_1 = \{1, 3, \ldots, l\} & s_2 = \{l+1, l+3, \ldots, L-1\} \\ s_3 = \{2, 4, \ldots, l-1\} & s_4 = \{l+2, l+4, \ldots, L\} \end{array} \tag{3.19}$$

and for regime 2^+:

$$t_1 = \{1, 3, \ldots, l-1\} \qquad t_2 = \{l+2, l+4, \ldots, L-1\}$$

$$t_3 = \{2, 4, \ldots, l\} \qquad t_4 = \{l+1, l+3, \ldots, L\} \tag{3.20}$$

where l is given by

$$l = \begin{cases} 2\left\lfloor \dfrac{L-1}{4} \right\rfloor + 1 & \text{regime } 1^+ \\ 2\left\lfloor \dfrac{L+1}{4} \right\rfloor & \text{regime } 2^+. \end{cases} \tag{3.21}$$

Then our forms for the solutions are

$$X_m^{abc}(q) = q^{mH(b,b,c)} \begin{cases} \mathcal{F}_m^{b,0}(b) - \mathcal{F}_m^{-b,0}(-b) & b \in s_1;\, b \in s_4 \text{ and } c = b+1 \\ \mathcal{F}_m^{b+1,0}(b) - \mathcal{F}_m^{-b-1,0}(-b) & b \in s_2;\, b \in s_3 \text{ and } c = b-1 \\ \mathcal{F}_m^{b+1,1}(b) - \mathcal{F}_m^{-b+1,1}(-b) & b \in s_3 \text{ and } c = b, b+1 \\ \mathcal{F}_m^{b+2,1}(b) - \mathcal{F}_m^{-b,1}(-b) & b \in s_4 \text{ and } c = b, b-1 \end{cases} \tag{3.22}$$

in regime 1^+. In regime 2^+, where, to avoid confusion, the solutions to (3.6) are called $Y_m^{abc}(q)$, we have

$$Y_m^{abc}(q) = q^{mH(b,b,c)} \begin{cases} \mathcal{G}_m^{b,0}(b) - \mathcal{G}_m^{-b,0}(-b) & b \in t_1;\, b \in t_4 \text{ and } c = b-1 \\ \mathcal{G}_m^{b-1,0}(b) - \mathcal{G}_m^{-b+1,0}(-b) & b \in t_2;\, b \in t_3 \text{ and } c = b+1 \\ \mathcal{G}_m^{b+1,1}(b) - \mathcal{G}_m^{-b+1,1}(-b) & b \in t_3 \setminus \{2\} \text{ and } c = b-1; \\ & \quad b \in t_3 \text{ and } c = b \\ \mathcal{G}_m^{3,0}(2) - \mathcal{G}_m^{-3,0}(-2) & b = 2 \text{ and } c = 1 \\ \mathcal{G}_m^{b,1}(b) - \mathcal{G}_m^{-b+2,1}(-b) & b \in t_4 \text{ and } c = b, b+1. \end{cases} \tag{3.23}$$

This representation of the solutions is preferable for three reasons. First, there is a pleasing symmetry between the structure of all elements, and direct analogy with the form (3.7) seen in other models. In turn, this makes the process of taking the thermodynamic limit ($m \to \infty$) simpler. Finally, performing the transformation $q \to 1/q$, which gives the solution in the remaining two regimes, is now straightforward even when m is finite. We now explore these last two ideas.

4. Regimes 3^+ and 4^+

The objects of interest in [7] were the order parameters (3.3) and (3.4). Hence it was only important to take $q \to 1/q$ in terms in $X_m^{abc}(q)$ which would then dominate once $m \to \infty$. Furthermore, it was necessary to do this only for the subset of configurations for which the values of b and c correspond to a ground-state phase. In the light of the recent work in which the bosonic polynomials for finite m have featured, a complete set of expressions for the configuration sums of the dilute A model in regimes 3^+ and 4^+ is pertinent. Of course, we wish to write them, including those already given in [7], in terms of trinomials.

In [7], the identity

$$\begin{bmatrix} m \\ k, l \end{bmatrix}_{1/q} = q^{k^2+l^2+kl-(k+l)m} \begin{bmatrix} m \\ k, l \end{bmatrix}_q \tag{4.1}$$

was used when taking $q \to 1/q$ in the regimes 1^+ and 2^+ solutions to find the configuration sums in regimes 4^+ and 3^+, respectively. The expressions which result resemble (3.8) and (3.9) but cannot be written in terms of the trinomials $\left[{m \atop A}\right]_2^{(p)}$. However, the literature [8] provides further q-analogues of the trinomial coefficients, among which is

$$T_p(m, A, q^{-1/2}) = q^{[A^2-m^2+p(m-A)]/2} \left[{m \atop A}\right]_2^{(p)} \tag{4.2}$$

or using the notation of Schilling [9]

$$\left.\left[{m \atop A}\right]_2^{(p)}\right|_{1/q} = q^{[A^2-m^2-p(A-m)]/2} T_p^{(2)}(m, A). \tag{4.3}$$

We are thus able to write

$$\begin{aligned} \mathcal{F}_m^{s,p}(b)|_{1/q} &= q^{[b(s+p)-a(2b-s+p)-m(m-p)]/2} \\ &\quad \times \sum_{j=-\infty}^{\infty} q^{L(L+1)j^2+j[La-(L+1)(2b-s+p)]} T_p^{(2)}(m, 2(L+1)j+a-b) \end{aligned} \tag{4.4}$$

$$\begin{aligned} \mathcal{G}_m^{r,p}(b)\big|_{1/q} &= q^{[b(r+p)-a(2b-r+p)-m(m-p)]/2} \\ &\quad \times \sum_{j=-\infty}^{\infty} q^{(L+1)(L+2)j^2+j[(L+2)a-(L+1)(2b-r+p)]} T_p^{(2)}(m, 2(L+1)j+a-b). \end{aligned} \tag{4.5}$$

It is convenient to define two further functions

$$\mathcal{H}_m^{s,p}(b) = q^{[(b-a)s-m(m-p)]/2} \sum_{j=-\infty}^{\infty} q^{L(L+1)j^2-j[(L+1)s-La]} T_p^{(2)}(m, 2(L+1)j+a-b) \tag{4.6}$$

$$\mathcal{I}_m^{r,p}(b) = q^{[(b-a)r-m(m-p)]/2} \sum_{j=-\infty}^{\infty} q^{(L+1)(L+2)j^2+j[(L+2)a-(L+1)r]} T_p^{(2)}(m, 2(L+1)j+a-b) \tag{4.7}$$

which should be compared with (3.16) and (3.12), respectively.

To obtain the configuration sums for regime 3^+ from those in 2^+ we replace q with $1/q$ in the five expressions from (3.15), though of course in their $\mathcal{G}$ form, giving

$$\begin{array}{ll} \text{(i)} & \mathcal{I}_m^{b,0}(b) - \mathcal{I}_m^{-b,0}(-b) \\ \text{(ii)} & \mathcal{I}_m^{b+1,0}(b) - \mathcal{I}_m^{-(b+1),0}(-b) \\ \text{(iii)} & \mathcal{I}_m^{b,1}(b) - \mathcal{I}_m^{-b,1}(-b) \\ \text{(iii*)} & \mathcal{I}_m^{1,0}(2) - \mathcal{I}_m^{-1,0}(-2) \\ \text{(iv)} & \mathcal{I}_m^{(b+1),1}(b) - \mathcal{I}_m^{-(b+1),1}(-b). \end{array} \tag{4.8}$$

Finally, the four expressions we need to write $X_m^{abc}(q)$ for regime 4^+ are

$$\begin{array}{ll} \text{(i)} & \mathcal{H}_m^{b,0}(b) - \mathcal{H}_m^{-b,0}(-b) \\ \text{(ii)} & \mathcal{H}_m^{b-1,0}(b) - \mathcal{H}_m^{-(b-1),0}(-b) \\ \text{(iii)} & \mathcal{H}_m^{(b-1),1}(b) - \mathcal{H}_m^{-(b-1),1}(-b) \\ \text{(iv)} & \mathcal{H}_m^{b,1}(b) - \mathcal{H}_m^{-b,1}(-b). \end{array} \tag{4.9}$$

5. Virasoro characters

Like those of other solvable models, in the thermodynamic ($m \to \infty$) limit, apart from some prefactors which cancel in the order parameters (3.3) and (3.4), the ground-state configuration sums of the dilute A model coincide with characters of the associated conformal algebra. In regimes 1^+ and 2^+ this is the unitary minimal series $M(p, p+1)$, and in the other two regimes the product of this series with the Ising model, $M(3, 4)$. The Rocha–Caridi form for the Virasoro character of $M(p, p+1)$, normalized as in [11], is

$$\chi_{r,s}^{(p+1)}(q) = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \{q^{j(p(p+1)j+(p+1)r-ps)} - q^{(jp+r)(j(p+1)+s)}\}. \tag{5.1}$$

It was established in [7] that for the ferromagnetic ground states†

$$\begin{array}{lll} \text{regime } 1^+ & X_\infty^{abb}(q) \sim \chi_{a,s}^{(L+2)}(q) & s = 1, 3, \ldots, L \\ \text{regime } 2^+ & X_\infty^{abb}(q) \sim \chi_{r,a}^{(L+1)}(q) & r = 1, 3, \ldots, L-2 \\ \text{regime } 3^+ & X_\infty^{abb}(q) \sim \chi_{1/16}(q)\chi_{a,s}^{(L+2)}(q) & s = 2, 4, \ldots, L+1 \\ \text{regime } 4^+ & X_\infty^{abb}(q) \sim \chi_{1/16}(q)\chi_{r,a}^{(L+1)}(q) & r = 2, 4, \ldots, L-1 \end{array} \tag{5.2}$$

and for the antiferromagnetic ground states

$$\begin{array}{lll} \text{regime } 3^+ & X_\infty^{abb+1}(q) \sim \chi_{\mu_a/2}(q)\chi_{a,s}^{(L+2)}(q) & s = 1, 3, \ldots, L \\ \text{regime } 4^+ & X_\infty^{abb+1}(q) \sim \chi_{\mu_a/2}(q)\chi_{r,a}^{(L+1)}(q) & r = 1, 3, \ldots, L-2. \end{array} \tag{5.3}$$

Here the Ising characters are

$$\chi_{\mu_a/2}(q) = \chi_{\mu_a+1,1}^{(4)}(q) \qquad \chi_{1/16}(q) = \chi_{1,2}^{(4)}(q) \tag{5.4}$$

and $\mu_a = 0, 1$ depends on the parity of a and b, and whether m is taken to infinity through odd or even values, a feature of antiferromagnetic phases.

These results can be obtained more simply than they were in [7], now that the configuration sums have all been written as the difference of two terms, each of which contains trinomials.

The asymptotic behaviour of the trinomials is [8]

$$\begin{aligned} &\lim_{m\to\infty} \begin{bmatrix} m \\ A \end{bmatrix}_2^{(0)} = \frac{1}{(q)_\infty} \\ &\lim_{m\to\infty} \begin{bmatrix} m \\ A \end{bmatrix}_2^{(1)} = \frac{1+q^A}{(q)_\infty} \\ &\lim_{\substack{(m-A)\text{ even} \\ m\to\infty}} T_0^{(2)}(m, A) = \frac{1}{2}\frac{(-q^{1/2}; q)_\infty + (q^{1/2}; q)_\infty}{(q)_\infty} = \frac{q^{1/48}\chi_0(q)}{(q)_\infty} \\ &\lim_{\substack{(m-A)\text{ odd} \\ m\to\infty}} T_0^{(2)}(m, A) = \frac{1}{2}\frac{(-q^{1/2}; q)_\infty - (q^{1/2}; q)_\infty}{(q)_\infty} = \frac{q^{1/48}\chi_{1/2}(q)}{(q)_\infty} \\ &\lim_{m\to\infty} T_1^2(m, A) = \frac{(-q; q)_\infty}{(q)_\infty} = \frac{q^{-1/24}\chi_{1/16}(q)}{(q)_\infty}. \end{aligned} \tag{5.5}$$

It is immediately apparent that, in the thermodynamic limit of (3.22), (3.23), (4.8) and (4.9), the trinomials always provide the prefactor $1/(q)_\infty$ which goes with the sum over j to build the appropriate character $\chi_{r,s}^{(p+1)}$. In regimes 3^+ and 4^+ they also provide the Ising character. It is then straightforward to observe that all configurations in (4.8) and (4.9),

† The characters used in [7] were not in the normalized form of (5.1).

including the non-ground-state ones, give the same characters as appear in (5.2) and (5.3), when the values of b are correctly chosen from (3.19) and (3.20). Finally, we observe that elements (iii) and (iv) in regimes 1^+ and 2^+, which do not correspond to ground states and hence were not treated in the thermodynamic limit in [7], give combinations of the characters from (5.2):

$$\begin{array}{lll} \text{regime } 1^+ & \text{(iii)} \sim q^{(a-b)a/2}\chi_{a,b}^{(L+2)}(q) + q^{(a-b)(a-2)/2}\chi_{a,b+2}^{(L+2)}(q) & b \in s_4 \\ & \text{(iv)} \sim q^{(a-b)(a+1)/2}\chi_{a,b-1}^{(L+2)}(q) + q^{(a-b)(a-1)/2}\chi_{a,b+1}^{(L+2)}(q) & b \in s_3 \\ \text{regime } 2^+ & \text{(iii)} \sim q^{(a-b)(a+1)/2}\chi_{b-1,a}^{(L+1)}(q) + q^{(a-b)(a-1)/2}\chi_{b+1,a}^{(L+1)}(q) & b \in t_3 \\ & \text{(iv)} \sim q^{(a-b)(a+2)/2}\chi_{b-2,a}^{(L+1)}(q) + q^{(a-b)a/2}\chi_{b,a}^{(L+1)}(q) & b \in t_4. \end{array} \tag{5.6}$$

6. Closing remarks

Bosonic polynomials, like those discussed above, have attracted interest because they provide one side of generalized Rogers–Ramanujan-type identities. This idea was explored for the dilute A_3 model in [10], with the notation of [7]. We are thus able to give alternate expressions to those in (13) of [10] for some elements:

$$\begin{aligned} Y_m^{122} &= q^m\{B_{3,1}(m,1,2) + q^{-1}B_{1,1}(m,1,0)\} \\ Y_m^{133} &= q^{m-3}B_{1,1}(m,1,3). \end{aligned} \tag{6.1}$$

In doing this, we have taken our expressions (iii) and (iv) and made use of the identity (2.11) to write all terms using $\begin{bmatrix} m \\ k \end{bmatrix}_2^{(0)}$; that this gives a simple expression is a special feature of $L = 3$.

Solutions to the recurrence relation (3.6) for the dilute A_L model have only appeared for L odd. Consequently, here we provide the natural 'language' for a future solution with L even. We have also made explicit the role of the q-trinomial coefficients, both $\begin{bmatrix} m \\ A \end{bmatrix}_2^{(p)}$ and $T_p^{(2)}(m, A)$, in the configuration sums of the dilute A model, giving our results for all regimes and phases.

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References

[1] Berkovich A 1994 *Nucl. Phys. B* **431** 315–48
[2] Warnaar S O and Pearce P A 1996 *Int. J. Mod. Phys.* A **11** 291–311
[3] Melzer E 1994 *Int. J. Mod. Phys.* A **9** 1115–36
[4] Warnaar S O 1996 *J. Stat. Phys.* **82** 657–85
[5] Andrews G E 1970 *Scripta Math.* **28** 297–305
[6] Warnaar S O, Nienhuis B and Seaton K A 1992 *Phys. Rev. Lett.* **69** 710–12
[7] Warnaar S O, Pearce P A, Seaton K A and Nienhuis B 1994 *J. Stat. Phys.* **74** 469–531
[8] Andrews G E and Baxter R J 1987 *J. Stat. Phys.* **47** 297–330
[9] Schilling A 1996 *Nucl. Phys.* B **467** 247–71
[10] Warnaar S O and Pearce P A 1994 *J. Phys. A: Math. Gen.* **27** L891–L897
[11] Warnaar S O 1996 *J. Stat. Phys.* **84** 49–83
[12] Warnaar S O 1996 Private communication

[13] Andrews G E 1976 *The Theory of Partitions* (Reading, MA: Addison-Wesley)
[14] Andrews G E 1990 *J. Amer. Math. Soc.* **3** 653–69
Andrews G E 1994 *Contemp. Math.* **166** 141–54
[15] Berkovich A, McCoy B M and Orrick W P 1996 *J. Stat. Phys.* **83** 795–837
[16] Warnaar S O 1997 *Commun. Math. Phys.* **184** 203–32
[17] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[18] Andrews G E, Baxter R J and Forrester P J 1984 *J. Stat. Phys.* **35** 193–266
[19] Pearce P A and Seaton K A 1989 *Ann. Phys.* **193** 326–66
[20] Scott L C 1996 *BSc (Hons) Project* School of Mathematics, LaTrobe University